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Yann Rébillé

CERMSEM

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Abstract

Since von Neuman and Morgenstern's (1944) contribution to game theory, the expected utility criterion has become the standard functional to evaluate risky prospects. Risky prospects are understood to be lotteries on a set of prizes. In which case a decision maker will recieve a precise prize with a given probability. A wide interest on imprecise object has been developped since Zadeh's (1978) contribution to artificial intelligence, through the use of possibility function (see Dubois Prade (1988)). In this setting a decision maker is uncertain about the precise features of the object he is dealing with. A first step has been readily made to rank imprecise objects in Rébillé (2005).
Our objective is to build a decision theory which deals with imprecise lotteries i.e. lotteries on imprecise prizes, a typical situation encountered in Ellsberg's experiment (1961).

Keywords: non-additive measures, possibility theory, Choquet integral, decision making.

*CERMSEM, Université Paris I, 106-112 Bd de l'Hôpital, 75647 Paris Cedex 13, France.

1 Introduction

Since von Neuman and Morgenstern's contribution to game theory ([23]), the expected utility criterion has become the standard functional to evaluate risky prospects. Risky prospects are understood to be lotteries on a set of prizes. In which case a decision maker will recieve a precise prize with a given probability. A wide interest on imprecise object has been developped since Zadeh's ([24]) contribution to artificial intelligence, through the use of possibility function (see [6]). In this setting a decision maker is uncertain about the precise features of the object he is dealing with. A first step has been readily made to rank imprecise objects in [17].
Our objective is to build a decision theory which deals with imprecise lotteries i.e. lotteries on imprecise prizes, a typical situation encountered in Ellsberg's experiment ([7]). This theory uses intensively belief functions which trace back to Choquet's ([2]) and Revuz ([18]) seminal works on non-additive set functions. However our approach does not contain any subjective treatement of beliefs as in Shafer's "Theory of Evidence" ([19]) or Smets "Transferable Belief Model" ([22]) and must be seen as a decision under (imprecise) risk.
The following section introduces the prerequiste material related to necessity measures, belief functions and the Choquet integral. The third section contains the central preferences representation theorem on imprecise lotteries. The fourth section extends the theory to imprecise probabilities. Finally we focus on imprecise risk and Ellsberg's paradox.

2 Notations, definitions

A set function $v : \mathcal{P}(\Omega) \mapsto [0,1]$ is a *belief function* if $v(\emptyset) = 0, v(\Omega) = 1$, v is monotone i.e. $\forall A \subset B \Rightarrow v(A) \leq v(B)$ and $\forall n \in \mathbb{N} \setminus \{0\}$, $\forall A_1, \ldots, A_n \subset \Omega$,

$$v(\cup_{k=1}^n A_k) \geq \sum_{\{I:\emptyset \neq I \subset \{1,\ldots,n\}\}} (-1)^{|I|+1} v(\cap_{k\in I} A_k)$$

where $|I|$ denotes the cardinal of I.
Let $Bel(\Omega)$ denote the set of belief functions on Ω.
For a belief function v defined on $\mathcal{P}(\Omega)$, there is a unique decomposition of v over elementary belief function given by the Möbius transform denoted m_v (see [1], [6], [8] [16], [19]) such that :

$$v = \sum_{B \neq \emptyset, \subset \Omega} m_v(B).u_B$$

where $\forall B \subset \Omega, \neq \emptyset, m_v(B) \geq 0$, $\sum_{B\neq\emptyset,\subset\Omega} m_v(B) = 1$ and u_B denotes an *elementary belief function* i.e. a *unanimity game* with support B defined by,

$$\forall A \subset \Omega, u_B(A) = \begin{cases} 1, & \text{if } B \subset A \\ 0, & \text{otherwise} \end{cases}$$

If $B = \{\omega\}$ for some $\omega \in \Omega$, $u_B = \delta_\omega$ is the Dirac measure at ω.
An important (proper) subset of $Bel(\Omega)$ denoted $Nec(\Omega)$ is the set of *necessity functions* (see [6]), i.e., belief functions which satisfy

$$\forall A, B \subset \Omega, v(A \cap B) = \text{Min } \{v(A), v(B)\}$$

A typical example of necessity is given by elementary belief functions.

Let $\mathcal{O}$ be a set of objects. A *lottery* on $\mathcal{O}$ is a simple probability of the form $(p_i, o_i)_{i \in I}$ for a finite set I and $\{p_i\} \subset (0,1]$ with $\sum_{i \in I} p_i = 1$, $o_i \in \mathcal{O}$ for $i \in I$. We denote the set of lotteries by $Lott(\mathcal{O})$. If a decision maker receives a lottery $(p_i, o_i)_{i \in I}$ it is understood that he will receive the object o_i with probability p_i.
A function $U : \mathcal{O} \longrightarrow \mathbb{R}$ is termed an *utility function.*
The *expected utility* of a lottery $(p_i, o_i)_{i \in I}$ is given by $\sum_{i \in I} p_i \; U(o_i)$.

A binary relation $\succeq$ on $\mathcal{X}$ (i.e. $\succeq \subset \mathcal{X}^2$) is said to be *complete* if for all $x, y \in \mathcal{X}$ we have $x \succeq y$ or $x \succeq y$, *transitive* if for all $x, y, z \in \mathcal{X}$ such that $x \succeq y$ and $y \succeq z$ then $x \succeq z$. A *weak order* $\succeq$ on $\mathcal{X}$ is a binary relation on $\mathcal{X}$ which is complete and transitive. As usual we shall write $x \succ y$ for $x \succeq y$ and not$(y \succeq x)$, $x \sim y$ for $x \succeq y$ and $y \succeq x$.
A functional $I : \mathcal{X} \longrightarrow \mathbb{R}$ *represents* the binary relation $\succeq$ if and only if for all x, y in $\mathcal{X}$ it holds

$$x \succeq y \iff I(x) \geq I(y)$$

The material introduced in the following subsections can be found in [15], [17].

2.1 Choquet integral

Let S be a nonempty set and $\mathcal{S}$ a non-empty family of subsets of S.
We do not require this family to be an algebra nor to contain the empty set $\emptyset$ or the unit S. Since our work will be devoted to the finite case we will assume from now on that S is finite, thus $\mathcal{S}$ too. A *set function* v on $\mathcal{S}$ is a real valued function $v : \mathcal{S} \longrightarrow \mathbb{R}$.
v is a *monotone* set function if for all $(A, B) \in \mathcal{S}^2, v(A) \leq \; v(B)$ whenever $A \subset B$. If $S \in \mathcal{S}$, then v is *normalized* as soon as $v(S) = 1$. If morever $\emptyset \in \mathcal{S}$ then v is a *capacity* as soon as $v(\emptyset) = 0$.
A function $X : S \longrightarrow [0,1]$ is said to be $\mathcal{S}$-*measurable* if for every $t \in (0,1)$, $\{s \in S : X(s) \geq t\}$ belongs to $\mathcal{S}$, as usual for sake of brevity we denote the weak upper level set by $\{X \geq t\}$. $B_{[0,1]}(S, \mathcal{S})$ will denote the set of $\mathcal{S}$-measurable functions taking values in $[0,1]$. This function space $B_{[0,1]}(S, \mathcal{S})$ might not be convex.
If v is a monotone set function defined on $\mathcal{S}$, and X belongs to $B_{[0,1]}(S, \mathcal{S})$, the *Choquet integral* of X with respect to v (see [2]), denoted $\int X dv$, is defined by

$$\int X dv = \int_0^1 v(\{X \geq t\}) dt$$

where the integral under consideration is an improper Riemann integral given by

$$\int_0^1 v(\{X \geq t\})dt = \lim_{\tau \downarrow 0} \int_\tau^{1-\tau} v(\{X \geq t\})dt$$

this quantity is well defined since the function $v(\{X \geq .\})$ has finite range on $(0,1)$.

By construction as soon as v is monotone the Choquet integral $\int(.)dv$ becomes a monotone functional i.e. $\forall X, Y \in B_{[0,1]}(S, \mathcal{S}), [X \geq Y] \Rightarrow [\int Xdv \geq \int Ydv]$.
Let be $X, Y \in B_{[0,1]}(S, \mathcal{S})$, X, Y are said to be *comonotonic* (*compatible*) if for all $(s,t) \in S^2$, $(X(s) - X(t))(Y(s) - Y(t)) \geq 0$.
A fundamental property of the Choquet integral is the one of *comonotonic affinity* (see also [4], [13], [21]): $\forall X, Y \in B_{[0,1]}(S, \mathcal{S})$, $\forall \alpha \in (0,1)$, if $\alpha.X + (1-\alpha).Y \in B_{[0,1]}(S, \mathcal{S})$ and X, Y are comonotonic then $\int \alpha.X + (1-\alpha).Ydv = \alpha. \int Xdv + (1-\alpha). \int Ydv$.

2.2 Necessity measures

Our concern in [17] was to rank necessity measures, therefore we exhibited a decision criterion based on the Choquet integral to evaluate any necessity. Let Ω be a finite non-empty set. Let S and $\mathcal{S}$ be defined by Ω and $\{A^u : A \neq \emptyset, A \subset \Omega\}$ where $A^u = \{B : A \subset B \subset \Omega\}$ (where A^u stands for the upset generated by A). These sets of subsets of Ω are known as *(principal) filters* (see [3]), we denote the set of filters by $\mathcal{F}(\Omega)$.

A family $\mathcal{F}$ of subsets of Ω is said to be a *filter* if,
(*i*) $\emptyset \notin \mathcal{F}, \Omega \in \mathcal{F}$,
(*ii*) $\forall A, B, [A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}]$,
(*iii*) $\forall A, B, [A \in \mathcal{F}, A \subset B \Rightarrow B \in \mathcal{F}]$.

Now since Ω is a finite set it comes that any filter is principal i.e. $\exists A \neq \emptyset \ / \ \mathcal{F} = A^u$. In particular if $A = \{\omega\}$ for some $\omega \in \Omega$ we retrieve an *ultrafilter*.

Proposition: *Let $v : \mathcal{P}(\Omega) \longrightarrow \mathbb{R}$ be a function. Then v is a necessity if and only if $v \in B_{[0,1]}(\Omega, \mathcal{F}(\Omega))$.*

Let us recall that the notion of comonotonicity for necessities has been readily introduced in [16] under the name of *agreement* for minitive set functions :
v, w agree if $\forall A, B \in \mathcal{P}(\Omega)$, $(v(A) - v(B))(w(A) - w(B)) \geq 0$.
A significant property can be obtained which states that $Nec(\Omega)$ is conditionally convex,

Proposition: *Let v, w be necessities and $\alpha \in (0,1)$ then v, w agree if and only if $\alpha.v + (1-\alpha).w$ is a necessity.*

As a consequence of these propositions, given a monotone set function β defined on $\mathcal{F}(\Omega)$ we can compute the Choquet integral of a necessity v with respect to β, $\int vd\beta$. And for all necessities v, w and $\alpha \in (0,1)$ if v, w agree then,

$$\int \alpha.v + (1-\alpha).w \; d\beta = \alpha \int vd\beta + (1-\alpha) \int wd\beta$$

This object was the criterion which was used to rank necessities in order to obtain a *weak integral representation*, that is for all necessities v, w:

$$v \succeq w \iff \int v d\beta \geq \int w d\beta$$

A further look at the Choquet integral of a necessity w.r.t. a monotone set function β could be better elicited if β was better explicitated. One can notice that the family $\mathcal{F}(\Omega)$ possesses a nice property, which is of being stable by intersection (since $A^u \cap B^u = (A \cup B)^u$ for $A, B \subset \Omega$).

This property suggests to consider the monotone set function β to be minitive i.e.

$$\forall A, B \subset \Omega, \neq \emptyset, \beta(A^u \cap B^u) = \min\{\beta(A^u), \beta(B^u)\}$$

Since we are dealing with a finite universe the values of β can be fully explicitated. For all $A \subset \Omega, \neq \emptyset$ it holds, $A^u = \cap_{\omega \in A}\{\omega\}^u$, letting $u(\omega) = \beta(\{\omega\}^u)$ we have,

$$\beta(A^u) = \min_{\omega \in A} u(\omega)$$

This last expression gives for the Choquet integral $\int(.)d\beta$ a *strong integral representation* which is:

$$\int v d\beta = \int u dv$$

The functionnals $Nec(\Omega) : v \mapsto U(v) = \int v d\beta, \int u dv \in \mathbb{R}$ are the utility functions we are seeking in order to achieve our decision theory on $Lott(Nec(\Omega))$.

3 Decision making over imprecise lotteries

In this section we provide an axiomatization of preferences that can be represented through an expected Choquet integral. We consider a decision maker who has to rank imprecise lotteries i.e. elements in $Lott(Nec(\Omega))$.
A typical element of $Lott(Nec(\Omega))$ will be written in the following manner $\mathbf{v} = (p_i, v_i)_{i \in I}$ where I a non-empty finite set, $\{v_i\}_{i \in I} \subset Nec(\Omega)$, $\{p_i\}_{i \in I} \subset (0, 1]$ and $\sum_{i \in I} p_i = 1$. Now we shall state some axioms that the binary relation $\succeq$ may fulfill.

(WO) $\succeq$ *is a weak order.*

(MON) Monotonicity: $\forall\{v_i\}_{i \in I}, \{w_i\}_{i \in I} \subset Nec(\Omega), \forall\{p_i\}_{i \in I} \subset (0, 1]$ *with* $\sum_{i \in I} p_i = 1, [\forall i \in I, v_i \geq w_i] \Rightarrow [(p_i, v_i)_{i \in I} \succeq (p_i, w_i)_{i \in I}]$.

A weaker version is the following

(WMON) Weak Monotonicity: $\forall v, w \in Nec(\Omega), v \geq w \Rightarrow (1, v) \succeq (1, w)$.

This axiom insures that the decision maker always prefers precise objects.

(CMP) Composition: $\forall\{v_i\}_{i \in I} \subset Nec(\Omega), v_1' \in Nec(\Omega), \forall\{p_i\}_{i \in I} \subset (0, 1]$ *with* $\sum_{i \in I} p_i = 1$, $\forall \alpha \in (0, 1)$, *if* v_1, v_1' *agree then* $(p_1, \alpha v_1 + (1-\alpha)v_1'; p_2, v_2; \ldots; p_n, v_n) \sim (\alpha p_1, v_1; (1-\alpha)p_1, v_1'; p_2, v_2; \ldots; p_n, v_n)$.

In particular if $I = \{1\}$ then the composition axiom states that $(1, \alpha.v + (1-\alpha).w) \sim (\alpha, v; (1-\alpha), w)$ as soon as v, w agree.

The composition axiom is of fundamental importance since it allows one to resume all information contained in an imprecise lottery into a belief function. Let $(p_i, v_i)_{i\in I}$ be an imprecise lottery, from the Moebius transform formula there are for all $i \in I$ some unique $\{(\alpha^i_j, A^i_j)\}_{j\in J_i}$ with $\alpha^i_j > 0$, $\sum_{j\in J_i} \alpha^i_j = 1$ and $\emptyset \neq A^i_j \subset \Omega$, $A^i_j \subsetneq A^i_{j'}$ for $j' < j$ such that $v_i = \sum_{j\in J_i} \alpha^i_j.u_{A^i_j}$. Applying the composition axiom successively gives

$$\begin{aligned}(p_1, v_1; \ldots; p_I, v_I) &= (p_1, \sum_{j\in J_1} \alpha^1_j.u_{A^1_j}; \ldots; p_I, \sum_{j\in J_I} \alpha^I_j.u_{A^I_j}) \\ &\sim (p_1\alpha^1_1, u_{A^1_1}; \ldots; p_1\alpha^1_{J_1}, u_{A^1_{J_1}}; \ldots; p_I\alpha^I_1, u_{A^I_1}; \ldots; p_1\alpha^I_{J_I}, u_{A^I_{J_I}})\end{aligned}$$

Now if we gather the A^i_j's which appear more then once we retrieve the Moebius transform of the belief function $v = \sum_{i\in I} p_i.v_i$.

It turns out that if $(p_i, v_i)_{i\in I}$, $(q_j, w_j)_{j\in J}$ are imprecise lotteries such that $\sum_{i\in I} p_i v_i = \sum_{j\in J} q_j w_j$ then since they admit the same Moebius transform they should be deemed equivalent.

(IND) Independence: $\forall\{v_i\}_{i\in I}$, $\{w_j\}_{j\in J}$, $\{x_k\}_{k\in K} \subset Nec(\Omega)$, $\forall\{p_i\}_{i\in I}$, $\{q_k\}_{j\in J}$, $\{r_k\}_{k\in K} \subset (0,1]$ *with* $\sum_{i\in I} p_i = \sum_{j\in J} q_j = \sum_{k\in K} r_k = 1$, *and* $\forall\alpha \in (0,1)$
if $(p_i, v_i)_{i\in I} \sim (q_j, w_j)_{j\in J}$ *then*

$$\begin{aligned}&(\alpha p_1, v_1; \ldots; \alpha p_I, v_I; (1-\alpha)r_1, x_1; \ldots; (1-\alpha)r_K, x_K) \\ \sim\ &(\alpha q_1, w_1; \ldots; \alpha q_J, w_J; (1-\alpha)r_1, x_1; \ldots; (1-\alpha)r_K, x_K)\end{aligned}$$

In particular if for all (i,j,k), v_i, w_j, x_k are Dirac measures we retrieve the standard independence axiom for lotteries, since we can identify ω with δ_ω.

(ARCH) $\succeq$ *is Archimedean:* $\forall v, w \in Nec(\Omega)$,
$[(1,v) \prec (1,w)] \Rightarrow [\exists\alpha \in (0,1)/\ (1,v) \prec (\alpha, w; 1-\alpha, u_\Omega)]$
and,
$[\exists\ \alpha \in (0,1)/\ (\alpha, w; 1-\alpha, u_\Omega) \prec (1,v) \preceq (1,w)] \Rightarrow [\exists\ \alpha' \in (\alpha,1)/\ (\alpha', w; 1-\alpha', u_\Omega) \preceq (1,v)]$

This archimedean axiom is rather inocuous since it does not invoque any continuity assumptions on the preferences as is it could be done on the infinite prize spaces $Nec(\Omega)$ (see [11]).

The last axiom is standard,

(NDEG) $\succeq$ *is not degenerate:* $\exists\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$ *such that* $\mathbf{v} \succ \mathbf{w}$

3.1 Weak integral representation of preferences

Theorem 3.1 *Let* $\succeq$ *be a binary relation on* $Lott(Nec(\Omega))$, *if* $\succeq$ *satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG) then there exists a monotone set function* $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that for all* $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$:

$$\mathbf{v} \succeq \mathbf{w} \iff \sum_{i \in I} p_i \int v_i d\beta \geq \sum_{j \in J} q_j \int w_j d\beta$$

Moreover there is an $\omega_1 \in \Omega$ *such that for all* v *in* $Lott(Nec(\Omega))$*,*

$$\mathbf{v} \sim (\sum_{i \in I} p_i \int v_i d\beta, \delta_{\omega_1}; 1 - \sum_{i \in I} p_i \int v_i d\beta, u_\Omega)$$

and $\beta(\{\omega_1\}^u) = 1, \beta(\{\Omega\}) = 0$.
Conversely, if the binary relation is represented by an Expectation of Choquet integrals with respect to a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that* $\beta(\{\Omega\}) = 0$ *and* $\beta(\{\omega_1\}^u) = 1$ *for some* $\omega_1 \in \Omega$ *then* $\succeq$ *satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG).*

We have achieved an expected utility representation $\sum_{i \in I} p_i U(v_i)$ of any imprecise lottery $\mathbf{v}$ where the utility function is computed through a Choquet integral $U(v) = \int v d\beta$.
In particular if $v_i = \delta_{\omega_i}$ for all $i \in I$ then the (precise) lottery $\mathbf{v}$ has for evaluation

$$\sum_{i \in I} p_i \int v_i d\beta = \sum_{i \in I} p_i \int \delta_{\omega_i} d\beta = \sum_{i \in I} p_i \beta(\{\omega_i\}^u) = \sum_{i \in I} p_i u(\omega_i)$$

the last expression being the standard utility expectation of the lottery $(p_i, \delta_{\omega_i})_{i \in I}$.

Proof: The converse is immediate.
In order to prove the first part of the theorem let us introduce an auxilliary relation $\succeq^*$ on $Nec(\Omega)$ defined in the following manner: $\forall v, w \in Nec(\Omega)$,

$$v \succeq^* w \iff (1, v) \succeq (1, w)$$

Now using Theorem 3.1 in [17] will give the desired result. Let us first recall the axioms involved, superscripted by a star.

(WO)* $\succeq^*$ *is a weak order.*

(MON) Monotonicity:* $\forall v, w \in Nec(\Omega), [v \geq w] \Rightarrow [v \succeq^* w]$.

(AGR) Agreement:* $\forall u, v, w, \in Nec(\Omega)$, $\forall \alpha \in (0,1)$ *if* u, w *agree and* v, w *agree then* $[u \sim^* v] \Rightarrow [\alpha.u + (1-\alpha)w \sim^* \alpha.v + (1-\alpha)w]$.

(ARCH)* $\succeq$ *is Archimedean:* $\forall v, w \in Nec(\Omega)$,
$[v \prec^* w] \Rightarrow [\exists \alpha \in (0,1)/\ v \prec^* \alpha.w + (1-\alpha).u_\Omega]$
and,
$[\exists \alpha \in (0,1)/\ \alpha.w + (1-\alpha).u_\Omega \prec^* v \preceq^* w] \Rightarrow [\exists \alpha' \in (\alpha, 1)/\ \alpha'.w + (1-\alpha').u_\Omega \preceq^* v]$

(NDEG)* $\succeq$ *is not degenerate:* $\exists v, w \in Nec(\Omega)$ *such that* $v \succ^* w$

Theorem: *Let* $\succeq^*$ *be a binary relation on* $Nec(\Omega)$*, if* $\succeq^*$ *satisfies (WO*), (MON*), (AGR*), (ARCH*), (NDEG*) then there exists a monotone set function* $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that for all* $v, w \in Nec(\Omega)$*:*

$$v \succeq^* w \iff \int v d\beta \geq \int w d\beta$$

Moreover there is an $\omega_1 \in \Omega$ *such that for all* v *in* $Nec(\Omega)$,

$$v \sim^* (\int v d\beta).\delta_{\omega_1} + (1 - \int v d\beta).u_\Omega$$

and $\beta(\{\omega_1\}^u) = 1, \beta(\{\Omega\}) = 0$.

Conversely, if the binary relation is represented by a Choquet integral with respect to a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that* $\beta(\{\Omega\}) = 0$ *and* $\beta(\{\omega_1\}^u) = 1$ *for some* $\omega_1 \in \Omega$ *then* $\succeq^*$ *satisfies (WO*), (MON*), (AGR*), (ARCH*), (NDEG*).*

Let us verify that the preference relation $\succeq^*$ derived from $\succeq$ satisfies (WO*), (MON*), (AGR*), (ARCH*), (NDEG*) as soon as $\succeq$ fulfills (WO), (WMON), (CMP), (IND), (ARCH), (NDEG).
(WO*), (MON*) are straightforward to check.
(ARCH*) follows from (ARCH) and (CMP) for $I = \{1\}$.
Let us check that (NDEG*) is satisfied. Assume on the contrary that $\forall v, w \in Nec(\Omega), v \sim^* w$ or equivalently $(1, v) \sim (1, w)$. Let $(p_i, v_i)_{i \in I}, (q_j, w_j)_{j \in J} \in$ *Lott*$(Nec(\Omega))$ then (IND) entails $(p_i, v_i)_{i \in I} \sim (1, v_1) \sim (1, w_1) \sim (q_j, w_j)_{j \in J}$, thus (NDEG) is not satisfied.
Finally let us check that (AGR*) is satisfied.
Let $u, v, w, \in Nec(\Omega), \alpha \in (0,1)$ where u, w and v, w agree with $u \sim^* v$. Since $(1, u) \sim (1, v)$ by (IND) we have $(\alpha, u; 1-\alpha, w) \sim (\alpha, v; 1-\alpha, w)$, and (WO),(CMP) entail $(1, \alpha u + (1-\alpha) w) \sim (\alpha, u; 1-\alpha, w) \sim (\alpha, v; 1-\alpha, w) \sim (1, \alpha v + (1-\alpha) w)$ that is $\alpha u + (1-\alpha) w \sim^* \alpha v + (1-\alpha) w$, so (AGR) is fulfilled.

From the first part of the theorem there exists a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ with $\beta(\{\Omega\}) = 0$ such that for all $v, w \in Nec(\Omega)$:

$$(1, v) \succeq (1, w) \iff \int v d\beta \geq \int w d\beta$$

and there is an $\omega_1 \in \Omega$ such that $\beta(\{\omega_1\}^u) = 1$ and such that for all v in $Nec(\Omega)$,

$$(1, v) \sim (1, (\int v d\beta).\delta_{\omega_1} + (1 - \int v d\beta).u_\Omega)$$

Now since $\delta_{\omega_1}, u_\Omega$ agree, (CMP) entails

$$(1, v) \sim (\int v d\beta, \delta_{\omega_1}; 1 - \int v d\beta, u_\Omega)$$

Let $\mathbf{v} = (p_i, v_i)_{i \in I} \in$ *Lott*$(Nec(\Omega))$, we shall prove by induction that

$$\mathbf{v} \sim (\sum_{i \in I} p_i \int v_i d\beta, \delta_{\omega_1}; 1 - \sum_{i \in I} p_i \int v_i d\beta, u_\Omega)$$

Without loss of generality let us assume that $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}\backslash\{0\}$. Since $\forall i \in I, (1, v_i) \sim (\int v_i d\beta, \delta_{\omega_1}; 1 - \int v_i d\beta, u_\Omega)$, (IND) entails for $i = 1$,

$$\mathbf{v} \sim (p_1 \int v_1 d\beta, \delta_{\omega_1}; p_1(1 - \int v_1 d\beta), u_\Omega; p_2, v_2, \ldots; p_n, v_n)$$

Assume for $k < n$ the following:

$$\mathbf{v} \sim (\sum_{i=1}^{k} p_i \int v_i d\beta, \delta_{\omega_1}; \sum_{i=1}^{k} p_i(1 - \int v_i d\beta), u_\Omega; p_{k+1}, v_{k+1}; \ldots; p_n, v_n)$$

since $(1, v_{k+1}) \sim (\int v_{k+1} d\beta, \delta_{\omega_1}; 1 - \int v_{k+1} d\beta, u_\Omega)$, (IND) entails

$$\begin{aligned}
\mathbf{v} &\sim (\sum_{i=1}^{k} p_i \int v_i d\beta, \delta_{\omega_1}; \sum_{i=1}^{k} p_i(1 - \int v_i d\beta), u_\Omega; p_{k+1}, v_{k+1}; \ldots; p_n, v_n) \\
&\sim (\sum_{i=1}^{k} p_i \int v_i d\beta, \delta_{\omega_1}; \sum_{i=1}^{k} p_i(1 - \int v_i d\beta), u_\Omega; \\
&\quad p_{k+1} \int v_{k+1} d\beta, \delta_{\omega_1}; p_{k+1}(1 - \int v_{k+1} d\beta), u_\Omega; \ldots; p_n, v_n) \\
&= (\sum_{i=1}^{k+1} p_i \int v_i d\beta, \delta_{\omega_1}; \sum_{i=1}^{k+1} p_i(1 - \int v_i d\beta), u_\Omega; p_{k+2}, v_{k+2}; \ldots; p_n, v_n)
\end{aligned}$$

We have proved that it holds also for $k+1$, now by induction we have the desired result.
It remains to prove that for all $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$:

$$\mathbf{v} \succeq \mathbf{w} \iff \sum_{i \in I} p_i \int v_i d\beta \geq \sum_{j \in J} q_j \int w_j d\beta$$

Let $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$, then $\mathbf{v} \succeq \mathbf{w} \iff$

$$\begin{array}{cl}
(\sum_{i \in I} p_i \int v_i d\beta, \delta_{\omega_1}; 1 - \sum_{i \in I} p_i \int v_i d\beta, u_\Omega) & \\
\succeq & \\
(\sum_{j \in J} q_j \int w_j d\beta, \delta_{\omega_1}; 1 - \sum_{j \in J} q_j \int w_j d\beta, u_\Omega) & \\
\iff & \text{, by (CMP)} \\
(1, (\sum_{i \in I} p_i \int v_i d\beta)\delta_{\omega_1} + (1 - \sum_{i \in I} p_i \int v_i d\beta) u_\Omega) & \\
\succeq & \\
(1, (\sum_{j \in J} q_j \int w_j d\beta)\delta_{\omega_1} + (1 - \sum_{j \in J} q_j \int w_j d\beta) u_\Omega) & \\
\iff & \text{, by (MON)} \\
(\sum_{i \in I} p_i \int v_i d\beta)\delta_{\omega_1} + (1 - \sum_{i \in I} p_i \int v_i d\beta) u_\Omega & \\
\geq & \\
(\sum_{j \in J} q_j \int w_j d\beta)\delta_{\omega_1} + (1 - \sum_{j \in J} q_j \int w_j d\beta) u_\Omega & \\
\iff & \\
\sum_{i \in I} p_i \int v_i d\beta \geq \sum_{j \in J} q_j \int w_j d\beta &
\end{array}$$

$\square$

The set function β can be seen as a *utility set function* u defined on $\mathcal{P}(\Omega)\setminus\{\emptyset\}$ where $\forall\ \emptyset \neq A \subset \Omega, u(A) = \beta(\{A\}^u)$. We provide now a justification for this terminology.
Let $\beta : \mathcal{F}(\Omega) \longrightarrow \mathbb{R}$ be a set function or putted differently $\beta \in \mathbb{R}^{\mathcal{F}(\Omega)}$. Since $\mathcal{F}(\Omega) = \{A^u : \emptyset \neq A \subset \Omega\}$ has cardinality $2^{|\Omega|} - 1$ we can see $\mathbb{R}^{\mathcal{F}(\Omega)}$ as a finite dimensional real vector space, therefore we shall exhibit a basis on this vector space and this will provide a natural decomposition for β.

Let $A \subset \Omega, \neq \emptyset$ and define the unanimity game with support A^u , u_{A^u} on $\mathcal{F}(\Omega)$ through

$$\forall B^u \in \mathcal{F}(\Omega), u_{A^u}(B^u) = \begin{cases} 1, & \text{if } A^u \subset B^u \text{ or equivalently } B \subset A \\ 0, & \text{otherwise} \end{cases}$$

We can prove that the family $\{u_{A^u}\}_{A\neq\emptyset}$ provides a linear basis for $\mathbb{R}^{\mathcal{F}(\Omega)}$.

Proof: Let $\{\lambda_A\}_{A\neq\emptyset} \subset \mathbb{R}$ such that $\sum_{A\neq\emptyset} \lambda_A u_{A^u} = 0$ we shall prove by induction that $\lambda_A = 0$ for all $A \neq \emptyset$.
We have $0 = \sum_{A\neq\emptyset} \lambda_A u_{A^u}(\Omega^u) = \lambda_\Omega$.
Assume for $1 < k \leq |\Omega|$ that $\lambda_A = 0$ for all A such that $k \leq |A| \leq |\Omega|$.
We shall prove that $\lambda_A = 0$ for all A such that $|A| = k - 1$.
Let A_0 such that $|A_0| = k - 1$.
We have $0 = \sum_{A\neq\emptyset} \lambda_A u_{A^u}(A_0^u) = \sum_{A:A_0\subset A} \lambda_A = \sum_{A:A_0\subset A, A\neq A_0} \lambda_A + \lambda_{A_0} = \lambda_{A_0}$, by induction hypothesis.
Now since there are $2^{|\Omega|} - 1$ elements in $\{u_{A^u} : A \neq \emptyset\}$, they form a linear basis.
$\square$

Now for any set function β there are unique coefficient $\{X_A\}_{A\neq\emptyset} \subset \mathbb{R}$ such that

$$\beta = \sum_{A\neq\emptyset} X_A u_{A^u}$$

The set function $u(B) = \beta(B^u) = \sum_{A:A\supset B} X_A$ can be seen seen as a *commonality function* in evidence theory (p.40 in [19]).
In particular given u, β the $\{X_A\}_{A\neq\emptyset} \subset \mathbb{R}$ can be retrieved through the inversion formulas: $\forall a_1, \ldots, a_K \in \Omega$

$$X_{\Omega\setminus\{a_1,\ldots,a_K\}} = \sum_{\emptyset\subset I\subset\{1,\ldots,K\}} (-1)^{K-|I|}\ u(\Omega \setminus \{a_i : i \in I\})$$

Proof: We shall procede by induction. For $K = 1$ it holds since $X_\Omega = u(\Omega)$.
Assume it holds for $1 \leq K < |\Omega|$ we shall prove that it holds for $K + 1$.
We have

$$\sum_{B:\Omega\setminus\{a_1,\ldots,a_{K+1}\}\subset,\neq B} X_B = \sum_{I:I\subset\neq\{a_1,\ldots,a_{K+1}\}} X_{\Omega\setminus\{a_i:i\in I\}}$$

$$= \sum_{I:I\subset\neq\{a_1,\dots,a_{K+1}\}} \sum_{\emptyset\subset J\subset I} (-1)^{|I|-|J|}\, u(\Omega\setminus\{a_i : i\in J\}), \text{ by induction hypothesis}$$

$$= \sum_{\emptyset\subset J\subset\neq\{a_1,\dots,a_{K+1}\}} \sum_{I:J\subset I\subset\neq\{a_1,\dots,a_{K+1}\}} (-1)^{|I|-|J|}\, u(\Omega\setminus\{a_i : i\in J\})$$

$$= \sum_{\emptyset\subset J\subset\neq\{a_1,\dots,a_{K+1}\}} u(\Omega\setminus\{a_i : i\in J\}) \sum_{I:J\subset I\subset\neq\{a_1,\dots,a_{K+1}\}} (-1)^{|I|-|J|}$$

but,

$$\sum_{I:J\subset I\subset\neq\{a_1,\dots,a_{K+1}\}} (-1)^{|I|-|J|} = \sum_{L:\emptyset\subset L\subset,\neq\{a_1,\dots,a_{K+1}\}\setminus J} (-1)^{|L|} = -(-1)^{|J|-(K+1)}$$

from Newton's binomial formula, so

$$\sum_{B:\Omega\setminus\{a_1,\dots,a_{K+1}\}\subset,\neq B} X_B = \sum_{\emptyset\subset J\subset\neq\{a_1,\dots,a_{K+1}\}} (-1)^{K+1-|J|}\, u(\Omega\setminus\{a_i : i\in J\})$$

and finally

$$X_{\Omega\setminus\{a_1,\dots,a_{K+1}\}} = u(\Omega\setminus\{a_1,\dots,a_{K+1}\}) - \sum_{B:\Omega\setminus\{a_1,\dots,a_{K+1}\}\subset,\neq B} X_B$$

$$= \sum_{\emptyset\subset J\subset\{1,\dots,K+1\}} (-1)^{K+1-|J|}\, u(\Omega\setminus\{a_i : i\in J\})$$

$\square$

We can now specify the conditions under which β becomes a monotone set function. A necessary and sufficient condition for that β to be monotone is that for all $B\subset\Omega,\neq\emptyset$ and for all $\omega\notin B, \beta(B^u)\geq\beta((B\cup\{\omega\})^u)$, since for $B,D\subset\Omega$ with $B\subset D$, there are $\omega_1,\dots,\omega_K\in\Omega$ such that $D=B\cup\{\omega_1,\dots,\omega_K\}$. So $B^u\supset(B\cup\{\omega_1\})^u\supset\ldots\supset D^u$. Now we can re-write the monotonicity condition through the linear combination:

$$\beta(B^u)\geq\beta((B\cup\{\omega\})^u)$$

$\iff$

$$\sum_{A\neq\emptyset} X_A u_{A^u}(B^u) \geq \sum_{A\neq\emptyset} X_A u_{A^u}((B\cup\{\omega\})^u)$$

$\iff$

$$\sum_{A:A\supset B} X_A \geq \sum_{A:A\supset B\cup\{\omega\}} X_A$$

$\iff$

$$\sum_{A:A\supset B,\omega\notin A} X_A \geq 0$$

A family $\{X_A\}_{A\neq\emptyset} \subset \mathbb{R}$ satisfying the previous condition will be termed a *basic utility assignement*. Let $v_{\mathbf{v}} = \sum_{i\in I} p_i v_i$ be the belief function associated to an imprecise lottery $\mathbf{v} = (p_i, v_i)_{i\in I}$. Theorem 3.1 can be restated in the following manner

Theorem 3.2 *Let $\succeq$ be a binary relation on Lott$(Nec(\Omega))$, if $\succeq$ satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG) then there exist basic utility assignement $\{X_A\}_{A\neq\emptyset} \subset \mathbb{R}$ with $X_\Omega = 0$ such that for all $\mathbf{v}, \mathbf{w} \in$ Lott$(Nec(\Omega))$:*

$$\mathbf{v} \succeq \mathbf{w} \iff \sum_{A\neq\emptyset} X_A v_{\mathbf{v}}(A) \geq \sum_{A\neq\emptyset} X_A v_{\mathbf{w}}(A)$$

Moreover there is an $\omega_1 \in \Omega$ such that for all v in Lott$(Nec(\Omega))$,

$$\mathbf{v} \sim (\sum_{A\neq\emptyset} X_A v_{\mathbf{v}}(A), \delta_{\omega_1}; 1 - \sum_{A\neq\emptyset} X_A v_{\mathbf{v}}(A), u_\Omega)$$

and $\sum_{A\neq\emptyset:\omega_1\in A} X_A = 1$.
Conversely, if the binary relation is represented by an Expecteted basic utility assignement then $\succeq$ satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG).

In particular if $\mathbf{v} = (p_i, \delta_{\omega_i})_{i\in I}$ we retrieve the standard expected utility functional. Since in this case $v_{\mathbf{v}} = \sum_{i\in I} p_i \delta_{\omega_i}$, and the evaluation becomes

$$\begin{aligned}
\sum_{A\neq\emptyset} X_A v_{\mathbf{v}}(A) &= \sum_{A\neq\emptyset} X_A \sum_{i\in I} p_i \delta_{\omega_i}(A) \\
&= \sum_{i\in I} p_i \sum_{A\neq\emptyset} X_A \delta_{\omega_i}(A) \\
&= \sum_{i\in I} p_i \sum_{A:\omega_i\in A} X_A \\
&= \sum_{i\in I} p_i u(\{\omega_i\})
\end{aligned}$$

Proof: From Theorem 3.1 there is a utility set function β such that any lottery $\mathbf{v}$ has for evaluation $\sum_{\imath\in I} p_i \int v_i d\beta$. The utility set function can be uniquely decomposed into a basic utility assignement $\{X_A\}_{A\neq\emptyset}$ through $\beta = \sum_{A\neq\emptyset} X_A u_{A^u}$. Now the evaluation becomes

$$\sum_{\imath\in I} p_i \int v_i d\beta = \sum_{\imath\in I} p_i \int v_i \ d\ [\sum_{A\neq\emptyset} X_A u_{A^u}]$$

$$= \sum_{\imath\in I} p_i \sum_{A\neq\emptyset} X_A \int v_i du_{A^u} = \sum_{\imath\in I} p_i \sum_{A\neq\emptyset} X_A \ \mathrm{Inf}\{v_i(B) : B \in A^u\}$$

$$= \sum_{\imath\in I} p_i \sum_{A\neq\emptyset} X_A v_i(A) = \sum_{A\neq\emptyset} X_A [\sum_{\imath\in I} p_i v_i(A)] = \sum_{A\neq\emptyset} X_A v_{\mathbf{v}}(A)$$

$\square$

One can interprete the rôle of the belief function $v_{\mathbf{v}}$ deduced from the imprecise lottery $\mathbf{v} = (p_i, v_i)_{\imath \in I}$ as follows. Assume we have to give a confidence measure for a set $A \subset \Omega$. Then the decision maker faces a random imprecise interval $(p_i, [v_i(A), v_i^d(A)])_{i \in I}$ which is deemed equivalent to the (deterministic) imprecise interval $[v_{\mathbf{v}}(A), v_{\mathbf{v}}^d(A)]$. The latter imprecise interval is seen as a mean imprecise interval since,

$$[v_{\mathbf{v}}(A), v_{\mathbf{v}}^d(A)] = [\sum_{\imath \in I} p_i v_i(A), \sum_{\imath \in I} p_i v_i^d(A)] = \sum_{\imath \in I} p_i [v_i(A), v_i^d(A)]$$

where the standard convex structure on $\mathcal{I}([0,1])$, the set of closed interval in $[0,1]$ is defined through $\alpha[a,b] + (1-\alpha)[c,d] = [\alpha a + (1-\alpha)c, \alpha b + (1-\alpha)d]$ for $\alpha \in [0,1], [a,b], [c,d] \in \mathcal{I}([0,1])$ (see [12]).
The final step that decision maker performs is to compute a weighted average of the mean imprecise interval through a subjective family of weights $\{X_A\}_{A \neq \emptyset}$.

We now obtain as a corollary a representation result involving a utility set function (see [14] for a similar functional but in a different setting),

Corollary 3.1 *Let $\succeq$ be a binary relation on $Lott(Nec(\Omega))$, if $\succeq$ satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG) then there a (antitone) utility set function assignement $u : \mathcal{P}(\Omega) \setminus \{\emptyset\} \longrightarrow [0,1]$ with $u(\Omega) = 0$ such that for all $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$:*

$$\mathbf{v} \succeq \mathbf{w} \iff \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) u(B) \geq \sum_{B \neq \emptyset} m_{v_{\mathbf{w}}}(B) u(B)$$

Moreover there is an $\omega_1 \in \Omega$ with $u(\{\omega_1\}) = 1$ such that for all $\mathbf{v}$ in $Lott(Nec(\Omega))$,

$$\mathbf{v} \sim (\sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) u(B), \delta_{\omega_1}; 1 - \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) u(B), u_{\Omega})$$

where $\{m_{v_{\mathbf{v}}}(B)\}_{B \neq \emptyset}$ are the Moebius coefficient of the belief function $v_{\mathbf{v}}$.
Conversely, if the binary relation is represented by an Expecteted utility set function then $\succeq$ satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG).

It becomes apparent that if $\mathbf{v} = (p_i, \delta_{\omega_i})_{i \in I}$ we retrieve the standard expected utility functional. Since in this case $v_{\mathbf{v}} = \sum_{i \in I} p_i \delta_{\omega_i}$ and the Moebius inverse is given by

$$\forall B \subset \Omega, \neq \emptyset, m_{v_{\mathbf{v}}}(B) = \begin{cases} p_i, & \text{if } B = \{\omega_i\} \\ 0, & \text{otherwise} \end{cases}$$

Proof: From Theorem 3.2 there is a basic utility assignement $\{X_A\}_{A \neq \emptyset} \subset \mathbb{R}$ with $X_\Omega = 0$ such that $\mathbf{v}$ has for evaluation

$$\sum_{A \neq \emptyset} X_A v_{\mathbf{v}}(A) = \sum_{A \neq \emptyset} X_A \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) u_B(A)$$

$$= \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) \sum_{A \neq \emptyset} X_A u_B(A) = \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) \sum_{A \neq \emptyset, B \subset A} X_A$$

$$= \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) \beta(B^u) = \sum_{B \neq \emptyset} m_{v_{\mathbf{v}}}(B) u(B)$$

□

3.2 Strong integral representation of preferences

In this subsection we specify the utility set function in order to obtain a utility function, for this we introduce an inclusion axiom which relates to the combination of imprecision (see [17] for a justification).

(INCL) Inclusion : For all $A, B \subset \Omega, \neq \emptyset$, $[(1, u_A) \succeq (1, u_B)] \Rightarrow [(1, u_{A \cup B}) \sim (1, u_B)]$.

We are now able to state our preference representation theorem,

Theorem 3.3 *Let* $\succeq$ *be a binary relation on* $Lott(Nec(\Omega))$, *if* $\succeq$ *satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG) and (INCL) then there exists a utility function* $u : \Omega \longrightarrow [0, 1]$ *such that for all* $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$*:*

$$\mathbf{v} \succeq \mathbf{w} \iff \int u \, dv_{\mathbf{v}} \geq \int u \, dv_{\mathbf{w}}$$

Moreover there are $\omega_1, \omega_0 \in \Omega$ *with* $u(\omega_1) = 1, u(\omega_0) = 0$ *such that for all* v *in* $Lott(Nec(\Omega))$,

$$\mathbf{v} \sim (\int u \, dv_{\mathbf{v}}, \delta_{\omega_1}; 1 - \int u \, dv_{\mathbf{v}}, u_\Omega)$$

Conversely, if the binary relation is represented by a utility function then $\succeq$ *satisfies (WO), (WMON), (CMP), (IND), (ARCH), (NDEG) and (INCL).*

Proof: The proof follows directly from Theorem 3.1 and Theorem 3.2 in [17] and the linearity of the Choquet integral with respect to set functions. □

One can see know that the utility function is in fact reconstructed through basic utility assignement in a simple way.
Let $\{\omega^k\}_{0 \leq k \leq |\Omega|}$ be an enumeration of Ω where $\omega_0 = \omega^0$ and $\omega_1 = \omega^{|\Omega|}$ such that $0 = u(\omega^0) \leq \ldots \leq u(\omega^k) \ldots \leq u(\omega^{|\Omega|}) = 1$.

The Choquet integral of u with respect to $\sum_{i \in I} p_i v_i$ is given by

$$\int u \, dv_{\mathbf{v}} = \sum_{k=0}^{|\Omega|} (u(\omega^k) - u(\omega^{k-1})) v_{\mathbf{v}}(\{\omega^k, \ldots, \omega^{|\Omega|}\}) = \sum_{A \neq \emptyset} X_A v_{\mathbf{v}}(A)$$

where

$$\forall A \neq \emptyset, \subset \Omega, X_A = \begin{cases} u(\omega^k) - u(\omega^{k-1}), & \text{if } A = \{\omega^k, \dots, \omega^{|\Omega|}\} \\ 0, & \text{otherwise} \end{cases}$$

4 Imprecise probabilities

Our aim now is to extend our results to a wider set then $Lott(Nec(\Omega))$, the set of imprecise probabilities. For this matter we will endow $Nec(\Omega)$ with a σ-algebra carefully chosen. Our construction follows the line of Revuz's ([18]) seminal contribution dealing with abstract spaces.
Let $\Sigma_{\leq}$ be the σ-algebra generated by the upper sections $[\nu, \rightarrow)$ for $\nu \in Nec(\Omega)$, where $[\nu, \rightarrow) = \{\nu' \in Nec(\Omega) : \nu \leq \nu'\}$. The objects that a decision maker will have to rank now are probabilities on the measurable space $(Nec(\Omega), \Sigma_{\leq})$, let $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ denote the set of probabilities.
Let $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ we can associate a set function v_μ in the following manner:

$$\forall A \subset \Omega, v_\mu(A) = \int_{Nec(\Omega)} \varphi_A(\nu) d\mu(\nu) = \int_0^1 \mu(\{\varphi_A \geq t\}) dt$$

where $\varphi_A(\nu) = \nu(A)$ for all $(A, \nu) \in \mathcal{P}(\Omega) \times Nec(\Omega)$.
In particular if $\mu = \mathbf{v}$ is an imprecise lottery, then $\forall A \subset \Omega$,

$$v_\mu(A) = \int_{Nec(\Omega)} \varphi_A(\nu) d(\sum_{i \in I} p_i \delta_{v_i})(\nu)$$

$$= \sum_{i \in I} p_i \int_{Nec(\Omega)} \varphi_A(\nu) d\delta_{v_i}(\nu) = \sum_{i \in I} p_i \varphi_A(v_i) = \sum_{i \in I} p_i v_i(A) = v_{\mathbf{v}}(A)$$

The set function v_μ is well defined since,

Lemma 4.1 *$\forall A \subset \Omega$, the mapping φ_A is $\Sigma_{\leq}$ -measurable (in the usual sense).*

Proof: Let $A \subset \Omega$, we have

$$\{\varphi_A \geq t\} = \begin{cases} [tu_A + (1-t)u_\Omega, \rightarrow), & \text{if } t \in [0, 1] \\ Nec(\Omega), & \text{if } t < 0 \\ \emptyset, & \text{if } t > 1 \end{cases}$$

$\square$

Proposition 4.1 *For all $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$, v_μ is a belief function on Ω.*

Proof: That v_μ is a capacity is clear.
Let $t \in (0, 1)$ and $A_1, \dots, A_n \subset \Omega$ and $I \subset \{1, \dots, n\}, \neq \emptyset$. We have,

$$\begin{aligned}\{\varphi_{\cap_I A_i} \geq t\} &= \{\nu \in Nec(\Omega) : \nu(\cap_I A_i) \geq t\}\\ &= \cap_I\{\nu \in Nec(\Omega) : \nu(A_i) \geq t\}, \text{ since the } \nu \text{ 's are necessities}\\ &= \cap_I\{\varphi_{A_i} \geq t\}\end{aligned}$$

From Poincaré's equality we have,

$$\begin{aligned}\sum_{\{I:\emptyset\neq I\subset\{1,\ldots,n\}\}}(-1)^{|I|+1}\mu(\cap_I\{\varphi_{A_i} \geq t\}) &= \mu(\cup_{i=1}^n\{\varphi_{A_i} \geq t\})\\ &\leq \mu(\{\varphi_{\cup_{i=1}^n A_i} \geq t\})\end{aligned}$$

since $\cup_{i=1}^n\{\varphi_{A_i} \geq t\} \subset \{\varphi_{\cup_{i=1}^n A_i} \geq t\}$.
Now integration on $(0,1)$ gives,

$$\sum_{\{I:\emptyset\neq I\subset\{1,\ldots,n\}\}}(-1)^{|I|+1}v_\mu(\cap_{i\in I}A_i) \leq v_\mu(\cup_{i=1}^n A_i)$$

$\square$

Conversely to any $v \in Bel(\Omega)$ we can associate an imprecise probability $\mu_v \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$:

$$\begin{aligned}\forall a \in \Sigma_{\leq}, \mu_v(a) &= \textstyle\sum_{B\neq\emptyset,\subset\Omega:u_B\in a} m_v(B)\\ \text{i.e. } \mu_v &= \textstyle\sum_{B\neq\emptyset,\subset\Omega} m_v(B)\ \delta_{u_B}\end{aligned}$$

where the $\{m_v(B)\}_{B\neq\emptyset,\subset\Omega}$ are the Moebius coefficient of v given by the Moebius decomposition.

One should pay attention that μ_v is not the standard Moebius transform of v. The standard Moebius transform of v denoted Φ_v is defined ([8]) on the finite σ-algebra $\mathcal{P}(\mathcal{P}(\Omega)/\{\emptyset\}$ through

$$\forall a \subset \mathcal{P}(\Omega)/\{\emptyset\},\ \Phi_v(a) = \sum_{B\neq\emptyset,\subset\Omega} m_v(B)\ \delta_B(a)$$

and

$$\forall A \subset \Omega,\ v(A) = \Phi_v(\{B : \emptyset \neq B \subset A\})$$

Our μ_v is defined on $\Sigma_{\leq}$ which is an infinite set.

Theorem 4.1 *For all $v \in \mathrm{Bel}(\Omega)$, $v_{\mu_v} = v$. Moreover $\mu_{(.)}$ is one to one and $v_{(.)}$ is onto.*

Proof: Let $v \in Bel(\Omega)$. By construction the mapping $v_{(.)}$ is affine, thus

$$v_{\mu_v} = v_{\sum_{B\neq\emptyset,\subset\Omega} m_v(B)\delta_{u_B}} = \sum_{B\neq\emptyset,\subset\Omega} m_v(B) v_{\delta_{u_B}}$$

It suffices to prove that $v_{\delta_{u_B}} = u_B$ for all $B \neq \emptyset, \subset \Omega$, which entails $v_{\mu_v} = \sum_{B\neq\emptyset,\subset\Omega} m_v(B)u_B = v$.
For $A \in \Omega$, $v_{\delta_{u_B}}(A) = \int_0^1 \delta_{u_B}(\{\varphi_A \geq t\})dt$. Let $t \in (0,1)$, now $\delta_{u_B}(\{\varphi_A \geq t\}) = 1$ if and only if $u_B \in \{\varphi_A \geq t\}$ if and only if $u_B(A) \geq t$, but since $u_B(A) \in \{0,1\}$ this equivalent to $u_B(A) = 1$. So $v_{\delta_{u_B}}(A) = u_B(A)$.
That $\mu_{(.)}$ is one to one and $v_{(.)}$ is onto are immediate consequences of $v_{\mu_{(.)}} = (.)$ on $Bel(\Omega)$. □

However we should remark that $v_{(.)}$ is not one to one. For instance let $\emptyset \neq A \subsetneq B \subset \Omega$ then for $\alpha \in (0,1)$ we have $v_{\alpha\delta_{u_A}+(1-\alpha)\delta_{u_B}} = v_{\delta_{\alpha u_A+(1-\alpha)u_B}} = \alpha u_A + (1-\alpha)u_B$. We can also remark that $\mu_{(.)}$ is not onto since for all $v \in Bel(\Omega)$, $\mu_v \in Lott(Nec(\Omega)) \subsetneq \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$.

4.1 Weak integral representation of preferences

Thanks to the material previousely introduced we can use the natural connection between imprecise probabilities and belief functions.

(MCMP) Moebius Composition: $\forall \mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq}), v_\mu = v_{\mu'} \Rightarrow \mu \sim \mu'$

or equivalently

$$\forall \mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq}), \mu \sim \mu_{v_\mu}$$

as soon as $\sim$ is transitive.
Indeed, let $\mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ such that $v_\mu = v_{\mu'}$ we have $\mu \sim \mu_{v_\mu} = \mu_{v_{\mu'}} \sim \mu'$. Conversely, let $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ then $(v_\mu) = v_{\mu_{(v_\mu)}}$ from Theorem 4.1, thus $\mu \sim \mu_{(v_\mu)}$.

The axiom (MCMP) is an extension of the composition axiom (CMP) to $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$. Let $\{v_i\}_{i\in I} \subset Nec(\Omega), v_1' \in Nec(\Omega)$ where v_1, v_1' agree, $\{p_i\}_{i\in I} \subset (0,1]$ with $\sum_{i\in I} p_i = 1$ and $\alpha \in (0,1)$, then

$$\begin{aligned} v_{p_1\delta_{\alpha v_1+(1-\alpha)v_1'}+p_2\delta_{v_2}+\ldots+p_n\delta_{v_n}} &= \alpha p_1 v_1 + (1-\alpha)p_1 v_1' + p_2 v_2 + \ldots + p_n v_n \\ &= v_{\alpha p_1\delta_{v_1}+(1-\alpha)p_1\delta_{v_1'}+p_2\delta_{v_2}+\ldots+p_n\delta_{v_n}} \end{aligned}$$

so by (MCMP),

$$(p_1, \alpha v_1 + (1-\alpha)v_1'; p_2, v_2; \ldots; p_n, v_n) \sim (\alpha p_1, v_1; (1-\alpha)p_1, v_1'; p_2, v_2; \ldots; p_n, v_n)$$

Theorem 4.2 *Let $\succeq$ be a binary relation on $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$, if $\sim$ is transitive and if $\succeq$ satisfies (WO), (WMON), (IND), (ARCH), (NDEG) and (MCMP) then there exists a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ such that for all $\mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$:*

$$\mu \succeq \mu' \iff I_\mu \geq I_{\mu'}$$

where

$$I_\mu = \int_{Nec(\Omega)} (\int \nu d\beta) \ d\mu(\nu)$$

Moreover there is an $\omega_1 \in \Omega$ *such that for all* μ *in* $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$,

$$\mu \sim (I_\mu, \delta_{\omega_1}; 1 - I_\mu, u_\Omega)$$

and $\beta(\{\omega_1\}^u) = 1, \beta(\{\Omega\}) = 0$.
Conversely, if the binary relation is represented by an Expectation of Choquet integrals with respect to a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ *such that* $\beta(\{\Omega\}) = 0$ *and* $\beta(\{\omega_1\}^u) = 1$ *for some* $\omega_1 \in \Omega$ *then* $\succeq$ *is a weak order on* $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ *which satisfies (WMON), (IND), (ARCH), (NDEG) and (MCMP).*

In particular we retrieve the functional obtained in Theorem 3.1, since for $\mu = \mathbf{v}$ we have:

$$\begin{aligned} I_\mu &= \int_{Nec(\Omega)} (\int \nu d\beta) \ d(\sum_{i \in I} p_i \delta_{v_i})(\nu) \\ &= \sum_{i \in I} p_i \int_{Nec(\Omega)} (\int \nu d\beta) \ d(\delta_{v_i})(\nu) = \sum_{i \in I} p_i \int v_i d\beta \end{aligned}$$

Proof: The converse is straight forward.
If we restrict the preference relation to the set of imprecise lotteries via Theorem 3.1 there exists a monotone set function $\beta : \mathcal{F}(\Omega) \longrightarrow [0,1]$ with $\beta(\{\Omega\}) = 0$ and $\omega_1 \in \Omega$ satisfying $\beta(\{\omega_1\}^u) = 1$, such that for all $\mathbf{v}, \mathbf{w} \in Lott(Nec(\Omega))$:

$$\mathbf{v} \succeq \mathbf{w} \iff \sum_{i \in I} p_i \int v_i d\beta \geq \sum_{j \in J} q_j \int w_j d\beta$$

and

$$\mathbf{v} \sim (\sum_{i \in I} p_i \int v_i d\beta, \delta_{\omega_1}; 1 - \sum_{i \in I} p_i \int v_i d\beta, u_\Omega)$$

Let $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$, by $(MCMP)$ we have $\mu \sim \mu_{v_\mu}$.
Now $v_\mu = \sum_{B \neq \emptyset, \subset \Omega} m_{v_\mu}(B) u_B$, thus $\mu_{v_\mu} = \sum_{B \neq \emptyset, \subset \Omega} m_{v_\mu}(B) \delta_{u_B}, \in Lott(Nec(\Omega))$.
So

$$\mu \sim (\sum_{B \neq \emptyset, \subset \Omega} m_{v_\mu}(B) \int u_B d\beta, \delta_{\omega_1}; 1 - \sum_{B \neq \emptyset, \subset \Omega} m_{v_\mu}(B) \int u_B d\beta, u_\Omega)$$

We can also compute the Moebius inverse of v_μ, let $B \subset \Omega$ we have

$$\begin{aligned} m_{v_\mu}(B) &= \sum_{T \subset B} (-1)^{|B|-|T|} v_\mu(T) \\ &= \sum_{T \subset B} (-1)^{|B|-|T|} \int_{Nec(\Omega)} \nu(T) d\mu(\nu) \\ &= \int_{Nec(\Omega)} \sum_{T \subset B} (-1)^{|B|-|T|} \nu(T) d\mu(\nu) \\ &= \int_{Nec(\Omega)} m_\nu(B) d\mu(\nu) \end{aligned}$$

and $m_{(.)}(B)$ is $\Sigma_{\leq}$-measurable as a finite sum of $(-1)^{|B|-|T|}(.)(T)$'s which are $\Sigma_{\leq}$-measurable. Now

$$\begin{aligned}
\sum_{B\neq\emptyset,\subset\Omega} m_{v_\mu}(B)(\int u_B d\beta) &= \sum_{B\neq\emptyset,\subset\Omega}[\int_{Nec(\Omega)} m_\nu(B)\ d\mu(\nu)](\int u_B d\beta) \\
&= \sum_{B\neq\emptyset,\subset\Omega}[\int_{Nec(\Omega)} m_\nu(B)(\int u_B d\beta)\ d\mu(\nu)] \\
&= \int_{Nec(\Omega)}[\sum_{B\neq\emptyset,\subset\Omega} m_\nu(B)(\int u_B d\beta)]\ d\mu(\nu)
\end{aligned}$$

This last formula can be rewritten

$$= \int_{Nec(\Omega)}[\sum_{B\neq\emptyset,\subset\Omega} m_\nu(B)\int u_B d\beta]\ d\mu(\nu) = \int_{Nec(\Omega)}[\int \nu d\beta]\ d\mu(\nu)$$

since for all $\nu \in Nec(\Omega)$ the set $\{B : m_\nu(B) > 0\}$ is finite and is a nested family of subsets, the elementary beliefs $\{u_B\}_B$ with $m_\nu(B) > 0$ agree thus

$$\sum_{B\neq\emptyset,\subset\Omega} m_\nu(B)\int u_B d\beta = \int \sum_{B\neq\emptyset,\subset\Omega} m_\nu(B) u_B d\beta = \int \nu d\beta$$

We have established that for all $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$

$$\mu \sim (I_\mu, \delta_{\omega_1}; 1 - I_\mu, u_\Omega)$$

where $I_\mu = \int_{Nec(\Omega)}(\int \nu d\beta)\ d\mu(\nu)$.
It remains to show that $I_{(.)}$ represents the preferences. Let μ, μ' be imprecise probabilities, then

$$\begin{aligned}
\mu \succeq \mu' &\iff (I_\mu, \delta_{\omega_1}; 1 - I_\mu, u_\Omega) \succeq (I_{\mu'}, \delta_{\omega_1}; 1 - I_{\mu'}, u_\Omega) \\
&\iff (1, I_\mu\delta_{\omega_1} + (1 - I_\mu)u_\Omega) \succeq (1, I_{\mu'}\delta_{\omega_1} + (1 - I_{\mu'})u_\Omega)\ ,\ \text{by (MCMP)} \\
&\iff I_\mu\delta_{\omega_1} + (1 - I_\mu)u_\Omega \geq I_{\mu'}\delta_{\omega_1} + (1 - I_{\mu'})u_\Omega\ ,\ \text{by (WMON)} \\
&\iff I_\mu \geq I_{\mu'}
\end{aligned}$$

□

The last theorem can be simply restated with basic utility assignement.

Theorem 4.3 *Let $\succeq$ be a binary relation on $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$, if $\sim$ is transitive and if $\succeq$ satisfies (WO), (WMON), (IND), (ARCH), (NDEG) and (MCMP) then there exist basic utility assignement $\{X_A\}_{A\neq\emptyset} \subset \mathbb{R}$ with $X_\Omega = 0$ such that for all $\mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$:*

$$\mu \succeq \mu' \iff I_\mu \geq I_{\mu'}$$

where

$$I_\mu = \sum_{A\neq\emptyset} X_A v_\mu(A)$$

Moreover there is an $\omega_1 \in \Omega$ *such that for all* μ *in* $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$,

$$\mu \sim (I_\mu, \delta_{\omega_1}; 1 - I_\mu, u_\Omega)$$

and $\sum_{A \neq \emptyset : \omega_1 \in A} X_A = 1$.
Conversely, if the binary relation is represented by an Expecteted basic utility assignement then $\succeq$ *is a weak order on* $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ *which satisfies (WMON), (IND), (ARCH), (NDEG) and (MCMP).*

Proof: From Theorem 4.2 and the decomposition of the set function β we have,

$$\begin{aligned} I_\mu &= \int_{Nec(\Omega)} (\int \nu d\beta)\ d\mu(\nu) \\ &= \int_{Nec(\Omega)} \sum_{A \neq \emptyset} X_A \nu(A)\ d\mu(\nu) \\ &= \sum_{A \neq \emptyset} X_A \int_{Nec(\Omega)} \nu(A)\ d\mu(\nu) \\ &= \sum_{A \neq \emptyset} X_A v_\mu(A) \end{aligned}$$

□

An interpretation similar than for imprecise lotteries can be given. Let $A \subset \Omega$. Then the decision maker faces a random imprecise interval $[v(A), v^d(A)]$ on $(Nec(\Omega), \Sigma_{\leq}, \mu)$ which is deemed equivalent to the imprecise interval $[\nu_\mu(A), \nu_\mu^d(A)]$. The latter imprecise interval is seen as an expected imprecise interval since,

$$[v_\mu(A), v_\mu^d(A)] = [\int_{Nec(\Omega)} \nu(A) d\mu, \int_{Nec(\Omega)} \nu^d(A) d\mu] = \int_{Nec(\Omega)} [\nu(A), \nu^d(A)] d\mu$$

where the mean of a random closed interval in $[0,1]$ with respect to a measure m on $(Y, \mathcal{B})$ is defined through

$$\int_Y [a(y), b(y)] dm(y) = [\int_Y a(y) dm(y), \int_Y b(y) dm(y)]$$

where a, b are $\mathcal{B}$-measurable functions taking values in $[0,1]$.

And similarely we can restate Corollary 3.1 with m_{v_μ} instead of $m_{v_{\mathbf{v}}}$.

Corollary 4.1 *Let* $\succeq$ *be a binary relation on* $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$*, if* $\sim$ *is transitive and if* $\succeq$ *satisfies (WO), (WMON), (IND), (ARCH), (NDEG) and (MCMP) then there a (antitone) utility set function assignement* $u : \mathcal{P}(\Omega) \setminus \{\emptyset\} \longrightarrow [0,1]$ *with* $u(\Omega) = 0$ *such that for all* $\mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$*:*

$$\mu \succeq \mu' \iff \sum_{B \neq \emptyset} m_{v_\mu}(B) u(B) \geq \sum_{B \neq \emptyset} m_{v_{\mu'}}(B) u(B)$$

Moreover there is an $\omega_1 \in \Omega$ with $u(\{\omega_1\}) = 1$ such that for all imprecise probability μ,

$$\mu \sim (\sum_{B\neq\emptyset} m_{v_\mu}(B)u(B), \delta_{\omega_1}; 1 - \sum_{B\neq\emptyset} m_{v_\mu}(B)u(B), u_\Omega)$$

where $\{m_{v_\mu}(B)\}_{B\neq\emptyset}$ are the Moebius coefficient of the belief function v_μ. Conversely, if the binary relation is represented by an Expecteted utility set function then $\succeq$ is a weak order on $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ which satisfies (WMON), (IND), (ARCH), (NDEG) and (MCMP).

4.2 Strong integral representation of preferences

Theorem 4.4 *Let $\succeq$ be a binary relation on $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$, if $\sim$ is transitive and $\succeq$ satisfies (WO), (WMON), (IND), (ARCH), (NDEG), (MCMP) and (INCL) then there exists a utility function $u : \Omega \longrightarrow [0, 1]$ such that for all $\mu, \mu' \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$:*

$$\mu \succeq \mu' \iff I_\mu \geq I_{\mu'}$$

where

$$I_\mu = \int u dv_\mu$$

Moreover there are $\omega_1, \omega_0 \in \Omega$ with $u(\omega_1) = 1, u(\omega_0) = 0$ such that for all μ in $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$,

$$\mu \sim (I_\mu, \delta_{\omega_1}; 1 - I_\mu, u_\Omega)$$

Conversely, if the binary relation is represented by a utility function then $\succeq$ is a weak order on $\mathcal{M}(Nec(\Omega), \Sigma_{\leq})$ which satisfies (WMON), (IND), (ARCH), (NDEG), (MCMP) and (INCL).

Proof: From Theorem 4.2 and Theorem 3.2 in [17], there exists a utility function $u : \Omega \longrightarrow [0, 1]$ where $u(\omega_1) = 1, u(\omega_0) = 0$ for some $\omega_1, \omega_0 \in \Omega$, such that for all $\mu \in \mathcal{M}(Nec(\Omega), \Sigma_{\leq})$

$$I_\mu = \int_{Nec(\Omega)} (\int u d\nu)\ d\mu(\nu)$$

Now since u has finite range, there are some $\alpha_i > 0$ with $\sum_{i=1}^n \alpha_i = 1$ and some $A_i \subset \Omega$ with $\omega_1 \in A_1 \subsetneq \ldots \subsetneq A_n \not\ni \omega_0$ such that $u = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$, where $\mathbb{1}_A$ denotes the indicator function of the set A. We have,

$$\int_{Nec(\Omega)} (\int u d\nu)\ d\mu(\nu) = \int_{Nec(\Omega)} (\int \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i} d\nu)\ d\mu(\nu)$$

$$= \int_{Nec(\Omega)} \sum_{i=1}^n \alpha_i \nu(A_i)\ d\mu(\nu) = \sum_{i=1}^n \alpha_i \int_{Nec(\Omega)} \nu(A_i)\ d\mu(\nu)$$

$$= \sum_{i=1}^{n} \alpha_i v_\mu(A_i) = \int u dv_\mu$$

□

5 Applications

5.1 Imprecise risk

Let S be a non-empty finite set, the set of states of the world and $\mathcal{C}$ be the set of consequences, which is also finite. A member of $\mathcal{C}^S$ is termed an *act*. Let $\Omega = S \times \mathcal{C}$. This set is clearly finite.
The decision maker has to rank acts X for any given information structure $(S, \mathcal{P}(S), v)$ where $v \in Bel(\Omega)$. A couple (X, v) is termed a *partially probabilized act*.
Now to a belief function v and an act X we can associate through the Moebius coefficient an imprecise lottery,

$$(m_v(B), u_{\{(s,X(s)):\ s\in B\}})_{B:m_v(B)>0}$$

If the sets B such that $m_v(B) > 0$ are singletons we retrieve a standard probability distribution since in that case any probability distribution $(P(\{s\}), X(s))_{s\in S}$ can be identified through

$$(P(\{s\}), \delta_{(s,X(s))})_{s\in S} = (P(\{s\}), u_{\{(s,X(s))\}})_{s\in S}$$

since $m_P(\{s\}) = P(\{s\})$ for all $s \in S$ and $m_P(B) = 0$ if B is not a singleton.

We consider a decision maker who has a preference relation $\succeq$ on $Lott(Nec(\Omega))$. Let us assume that its preferences satisfy the axioms of Theorem 3.3, then there exists a utility function $U : \Omega \longrightarrow [0, 1]$ such that any partially probabilized act (X, v) is evaluated through

$$\sum_{B\neq\emptyset,\subset\Omega} m_v(B) \int U\ du_{\{(s,X(s)):s\in B\}} = \sum_{B\neq\emptyset,\subset\Omega} m_v(B)\ \mathrm{Min}\{U(s, X(s)) : s \in B\}$$

If furthermore the preferences of the decision maker are *state-independent* i.e. satisfy $\forall s, s' \in S,\ c \in \mathcal{C}, (c_S, \delta_s) \sim (c_S, \delta_{s'})$ where $c_S(s) = c$ for all $s \in S$; then the utility function U becomes *state-independent* i.e. $U(s, c) = U(s', c)$ for all $s, s' \in S, c \in \mathcal{C}$ and the evaluation becomes

$$\sum_{B\neq\emptyset,\subset\Omega} m_v(B) \int U\ du_{\{(s,X(s)):s\in B\}} = \sum_{B\neq\emptyset,\subset\Omega} m_v(B)\ \mathrm{Min}\{U(X(s)) : s \in B\}$$

$$= \int U(X)\ d\,[\sum_{B\neq\emptyset,\subset\Omega} m_v(B) u_B] = \int U(X)\ dv$$

we retrieve the Choquet utility expectation of the act X with respect to the belief function v.
Particular cases are of interest, if $v = P$ is a probability defined on $(S, \mathcal{P}(S))$ then we get the standard utility expectation, $\int U(X)\ dP$.
If $v = u_S$ is the trivial belief which represents *complete ignorance* then we get the maxmin utility criterion, $\int U(X)\ du_S = \text{Min}\{U(X(s)) :\ s \in S\}$,

The Choquet expected criterion $\int U(.)\ d(.)$ possesses some nice monotonicity properties.
Properties: *Let X, Y be two acts. The following statements are equivalent:*
(i) $U(X(s)) \geq U(Y(s))$ for all $s \in S$,
(ii) For any information strucure (S, v) the partially probabilized act (X, v) is prefered to (Y, v).
Let (S, v), (S, w) be two information structures. The following statements are equivalent:
(iii) v is more informative *than w i.e. $v \geq w$,*
(iv) for all act X the partially probabilized act (X, v) is prefered to (X, w).

Proof: $((i) \Rightarrow (ii))$ and $((iii) \Rightarrow (iv))$ are immediate.
$((ii) \Rightarrow (i))$ follows from $U(Z(s)) = \int U(Z)\ dv$ when $v = \delta_{(s,Z(s))}$ for $Z = X, Y$.
$((iv) \Rightarrow (iii))$. Pick $c_0, c_1 \in \mathcal{C}$ such that $U(c_0) < U(c_1)$. Let $A \subset S$ and define $X_A = c_1 1\!\!1_A + c_0 1\!\!1_{A^c}$. We have, $\int U(X_A)\ dv = U(c_0) + v(A)(U(c_1) - U(c_0)) \geq U(c_0) + w(A)(U(c_1) - U(c_0)) = \int U(X_A)\ dw$ so $v(A) \geq w(A)$. □

Notice that a maximal element in $Bel(\Omega)$ according to the ordering relation more informative is precisely a probability since any probability is always maximal and that any belief function is always less informative than some probability i.e. for all $v \in Bel(\Omega)$ there exists a probability P such that $P \geq v$ ([20]).

Particular cases which use the notion of “informativeness” can be easely constructed (see [10]).
Let $\mathcal{A} \subset \mathcal{P}(S)$ be an algebra and P be a probability defined on $(S, \mathcal{A})$. Denote by P_* its inner extension to $\mathcal{P}(S)$ defined through

$$P_*(A) = \text{Max}\ \{P(B) : B \subset A, B \in \mathcal{A}\}$$

We have that P_* is a belief since $P_* = \sum_i P(A_i) u_{A_i}$ where $\{A_i\}_i$ is the partition given by the atoms of $\mathcal{A}$.

(1) Let P is a probability defined on $(S, \mathcal{P}(S))$ and $\mathcal{A}, \mathcal{B}$ are subalgebras, if $\mathcal{B} \subset \mathcal{A}$ then for all act X,

$$(X, P) \succeq (X, (P_{|\mathcal{A}})_*) \succeq (X, (P_{|\mathcal{B}})_*) \succeq (X, (P_{|\{\emptyset,S\}})_*)$$

since $P \geq (P_{|\mathcal{A}})_* \geq (P_{|\mathcal{B}})_* \geq (P_{|\{\emptyset,S\}})_* = u_S$, where $P_{|\mathcal{D}}$ denotes the restriction of P to $\mathcal{D}$ for $\mathcal{D} = \mathcal{A}, \mathcal{B}, \{\emptyset, S\}$.

(2) Let P be probability defined on the measurable space $(S, \mathcal{A})$. Consider a probabilty $\tilde{P}$ on $(S, \mathcal{P}(S))$ then $\tilde{P}$ is an extension of P to $\mathcal{P}(S)$ if and only if

$\tilde{P} \geq P_*$, that is to say if and only if $\tilde{P}$ is more informative than P. So for all act X,

$$(X, \tilde{P}) \succeq (X, P)$$

5.2 Ellsberg's paradox

We conclude this section with the wellknown Ellsberg's [7] paradox as an exemple of imprecise risk situation. The Ellsberg experiment consists in bets on the outcome of a single draw from an urn which contains 30 red balls and 60 black or yellow balls in an unknown proportion. A decision maker is given a choice to bet on whether the outcome of the draw to be red or to be black and another choice to bet on whether the outcome will be red or yellow or whether it will be black or yellow. In each case the decision maker wins 1 euro if the colour he bets upon does indeed show up; otherwise nothing is gained or lost.
Let $S = \{R, B, Y\}$ denote the states of nature, and $\mathcal{C} = \{0, 1\}$ the set of consequences.
Let $X_i : S \longrightarrow \mathcal{C}$ for $i = 1, \ldots, 4$ be

$$X_1(R) = 1, X_1(B) = 0, X_1(Y) = 0$$

$$X_2(R) = 0, X_2(B) = 1, X_2(Y) = 0$$

$$X_3(R) = 1, X_3(B) = 0, X_3(Y) = 1$$

$$X_4(R) = 0, X_4(B) = 1, X_4(Y) = 1$$

When ask to choose between bets X_1, X_2 most people decide for X_1, and when the same people are asked to choose between X_3, X_4 they decide for X_4. Our aim here is to show that this pattern of decision making can be rationalized through our decision model. Let us assume that $U(1) = 1 > 0 = U(0)$.
A natural way to represent the information structure available to the decision maker is to use an inner extension as presented in the previous section.
Consider the probability P defined on $\{\emptyset, \{R\}, \{B, Y\}\}$ with $P(\{R\}) = \frac{1}{3}$. With this information structure the decision maker knows that with probability $\frac{1}{3}$ that a red ball will be drawn out of the urn and with probability $\frac{2}{3}$ that a black or yellow ball will be drawn out of the urn, however he is completely ignorant about the probability that a black ball will be drawn. The information structure resumes to $P_* = \frac{1}{3}\delta_R + \frac{2}{3}u_{\{B,Y\}}$.

Now the evaluations of the X_i's are

$$\int U(X_1)\ dP_* = \frac{1}{3} > 0 = \int U(X_2)\ dP_*$$

and

$$\int U(X_3)\ dP_* = \frac{1}{3} < \frac{2}{3} = \int U(X_4)\ dP_*$$

It should be noticed that in this setting the informative structure P_* is not derived from the preferences of the decision maker in a subjective way as is done in non-addtive expected utility under uncertainty but is objectively given by the experimental conditions.

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